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Derivatives

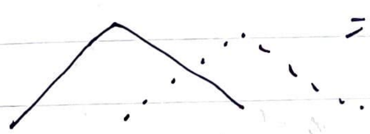
derivative of matrices

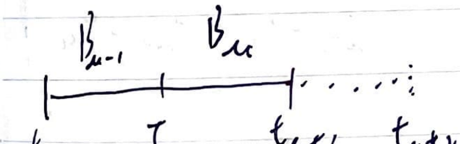
$$D(AB) = (DA)B + A(DB)$$

proof:
$$D(AB)_{ij} = D \sum_k a_{ik} b_{kj}$$
$$= \sum_k (D a_{ik}) b_{kj} + \sum_k a_{ik} (D b_{kj})$$
$$= (DA)B + A(DB)$$

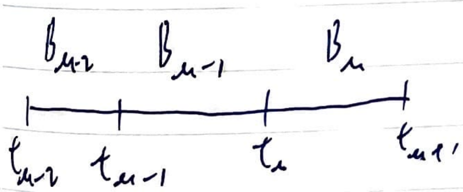
recall that $B_d(x) = \begin{pmatrix} B_{\mu-d,d} & B_{\mu-d+1,d} & \cdots & B_{\mu,d} \end{pmatrix} \quad . \; t \in [t_\mu, t_{\mu+1})$

$$= R_1(x) R_2(x) \cdots R_d(x)$$



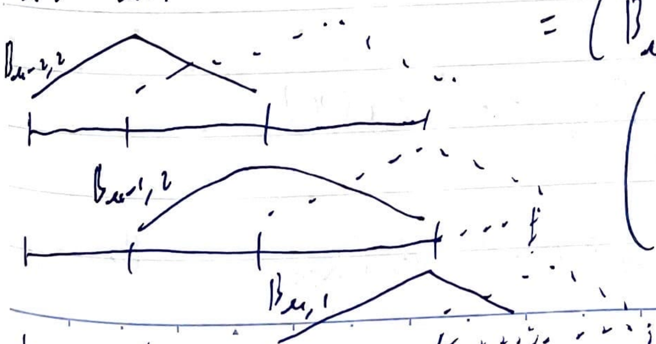


$$\begin{pmatrix} B_{\mu-1,1} & B_{\mu,1} \end{pmatrix} = \begin{pmatrix} \frac{t_{\mu+1}-x}{t_{\mu+1}-t_\mu} & \frac{x-t_\mu}{t_{\mu+1}-t_\mu} \end{pmatrix}$$



$$\begin{pmatrix} B_{\mu-2,2} & B_{\mu-1,2} & B_{\mu,2} \end{pmatrix}$$

$$= \begin{pmatrix} B_{\mu-1,1} & B_{\mu,1} \end{pmatrix} \begin{pmatrix} \frac{t_{\mu+1}-x}{t_{\mu+1}-t_{\mu-1}} & \frac{x-t_{\mu-1}}{t_{\mu+1}-t_{\mu-1}} & 0 \\ 0 & \frac{t_{\mu+2}-x}{t_{\mu+2}-t_\mu} & \frac{x-t_\mu}{t_{\mu+2}-t_\mu} \end{pmatrix}$$



$$\therefore R_d(x) = \begin{bmatrix} \frac{t_{\mu+1}-x}{t_{\mu+1}-t_{\mu+1-d}} & \frac{x-t_{\mu+1-d}}{t_{\mu+1}-t_{\mu+1-d}} & 0 & \cdots & 0 \\ 0 & \frac{t_{\mu+2}-x}{t_{\mu+2}-t_{\mu+2-d}} & \frac{x-t_{\mu+2-d}}{t_{\mu+2}-t_{\mu+2-d}} & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \frac{t_{\mu+d}-x}{t_{\mu+d}-t_{\mu}} & \frac{x-t_{\mu}}{t_{\mu+d}-t_{\mu}} \end{bmatrix}$$
$(d, d+1)$

recall also that $R_{d-1}(z) R_d(x) = R_{d-1}(x) R_d(z)$

$$DR_d(x) = \begin{bmatrix} \frac{-1}{t_{\mu+1}-t_{\mu+1-d}} & \frac{1}{t_{\mu+1}-t_{\mu+1-d}} & 0 & \cdots & 0 \\ 0 & \frac{-1}{t_{\mu+1}-t_{\mu+1-d}} & \frac{1}{t_{\mu+1}-t_{\mu+1-d}} & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \frac{-1}{t_{\mu+d}-t_{\mu}} & \frac{1}{t_{\mu+d}-t_{\mu}} \end{bmatrix}$$

taking derivative of $R_{d-1}(z) R_d(x) = R_{d-1}(x) R_d(z)$ w.r.t. z

$$DR_{d-1} R(x) = R_{d-1}(x) DR_d$$

furthermore,

$$DB_d(x) = \sum_{k=1}^{d} R_1(x) R_2(x) \cdots DR_k \cdots R_d(x)$$

by shifting the $DR_k R_{k+1}(x) = R_k(x) DR_{k+1}$ to the last term, we get

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$$\therefore \boxed{DB_d(x) = d\, B_{d-1}(x)\, DR_d}$$

$$D^2 B_d(x) = d\, D\left[B_{d-1}(x)\, DR_d \right]$$

$$= d\, DB_{d-1}(x)\, DR_d \qquad \because D(DR_d) = 0$$

$$= d\, B_{d-2}(x)\, DR_{d-1}\, DR_d \cdot (d-1)$$

$$\therefore \boxed{D^r B_d(x) = \frac{d!}{(d-r)!} B_{d-r}(x)\, DR_{d-r+1} \cdots DR_d} \qquad d! = 1 \cdot 2 \cdots d$$

$$\therefore DB_d(x) = \left(DB_{\mu-d,d}\ \ DB_{\mu-d+1,d}\ \ DB_{\mu-d+1,d} \cdots DB_{\mu,d} \right)$$

$$= d \left(B_{\mu-d+1,d-1}\ \ B_{\mu-d+2,d-1} \cdots B_{\mu,d-1} \right) DR_d$$

$$= d \left(B_{\mu-d+1,d-1} \cdots B_{\mu,d-1} \right) \begin{pmatrix} \frac{-1}{t_{\mu+1}-t_{\mu+1-d}} & \frac{1}{t_{\mu+1}-t_{\mu+1-d}} & & \\ & \ddots & & \\ & & \frac{-1}{t_{\mu+d}-t_{\mu}} & \frac{1}{t_{\mu+d}-t_{\mu}} \end{pmatrix}$$

$$= d \Bigg(\frac{-B_{\mu-d+1,d-1}}{t_{\mu+1}-t_{\mu+1-d}} \quad \frac{B_{\mu-d+1,d-1}}{t_{\mu+1}-t_{\mu+1-d}} - \frac{B_{\mu-d+2,d-1}}{t_{\mu+2}-t_{\mu+2-d}}$$

$$\frac{B_{\mu-d+2,d-1}}{t_{\mu+2}-t_{\mu+2-d}} - \frac{B_{\mu-d+3,d-1}}{t_{\mu+3}-t_{\mu+3-d}} \cdots \frac{B_{\mu-d+k,d-1}}{t_{\mu+k}-t_{\mu+k-d}} - \frac{B_{\mu-d+k+1,d-1}}{t_{\mu+k+1}-t_{\mu+k+1-d}}$$

$$\cdots \frac{B_{\mu,d-1}}{t_{\mu+d}-t_{\mu}} \Bigg)$$

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by comparing the terms on both side,

$$\boxed{DB_{j,d} = d\left(\frac{B_{j,d-1}}{t_{j+d}-t_j} - \frac{B_{j+1,d-1}}{t_{j+1+d}-t_{j+1}}\right)}$$

$$DB_d = d R_1 \cdots R_{d-1} DR_d$$

$$= d D(R_1 \cdots R_d) - d D(R_1 \cdots R_{d-1}) R_d$$

$$\therefore (d-1) DB_d = d DB_{d-1} R_d$$

$$= d\left(DB_{\mu-d+1,d-1}\ DB_{\mu-d+2,d-1} \cdots DB_{\mu,d-1}\right)\begin{pmatrix} \frac{t_{\mu+1}-x}{t_{\mu+1}-t_{\mu+1-d}} & \frac{x-t_{\mu+1-d}}{t_{\mu+1}-t_{\mu+1-d}} & & 0 \\ & & \ddots & \\ & 0 & \frac{t_{\mu+d}-x}{t_{\mu+d}-t_\mu} & \frac{x-t_\mu}{t_{\mu+d}-t_\mu} \end{pmatrix}$$

$$\therefore DB_d = \frac{d}{d-1}\left(DB_{\mu-d+1,d-1} \cdots DB_{\mu,d-1}\right) R_d$$

$$= \left(DB_{\mu-d,d}\ DB_{\mu-d+1,d} \cdots DB_{\mu,d}\right)$$

$$= \frac{d}{d-1}\left(\frac{t_{\mu+1}-x}{t_{\mu+1}-t_{\mu+1-d}} DB_{\mu+1-d,d-1}\quad \frac{x-t_{\mu+1-d}}{t_{\mu+1}-t_{\mu+1-d}} DB_{\mu+1-d,d-1} + \frac{t_{\mu+2}-x}{t_{\mu+2}-t_{\mu+2-d}} DB_{\mu+2-d,d-1}\right.$$

$$\cdots \frac{x-t_{\mu+k-d}}{t_{\mu+k}-t_{\mu+k-d}} DB_{\mu+k-d,d-1} + \frac{t_{\mu+k+1}-x}{t_{\mu+k+1}-t_{\mu+k+1-d}} DB_{\mu+k+1-d,d-1}$$

$$\left.\cdots \frac{x-t_\mu}{t_{\mu+d}-t_\mu} DB_{\mu,d-1}\right)$$

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$$\therefore \boxed{DB_{j,d} = \frac{d}{d-1}\left(\frac{x-t_j}{t_j - t_{j-d}} DB_{j,d-1} + \frac{t_{j+1}-x}{t_{j+1}-t_{j+1-d}} DB_{j+1,d-1}\right)}$$

Jump and smoothness

recall that

$$B_{j,d,\vec{t}}(x) := \frac{x-t_j}{t_{j+d}-t_j} B_{j,d-1,\vec{t}}(x) + \frac{t_{j+d+1}-x}{t_{j+d+1}-t_{j+1}} B_{j+1,d-1,\vec{t}}(x)$$

$$B_{j,0,\vec{t}}(x) = \begin{cases} 1 & t_j \le x < t_{j+1} \\ 0 & \text{else} \end{cases}$$

$$J_x(B) = \lim_{x \to x^+} B - \lim_{x \to x^-} B$$

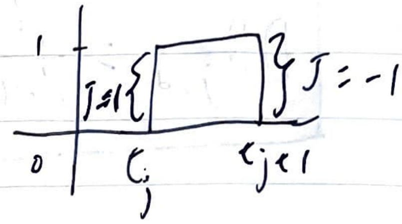
$\therefore \cancel{J_x(B_{j,d,\vec{t}}(x))}$

$$\therefore J_x(B_{j,d}) = \lim_{x \to x^+}\left(\frac{x-t_j}{t_{j+d}-t_j} B_{j,d-1}\right) - \lim_{x \to x^-}\left(\frac{x-t_j}{t_{j+d}-t_j} B_{j,d-1}\right)$$

$$+ \lim_{x \to x^+}\left(\frac{t_{j+d+1}-x}{t_{j+d+1}-t_{j+1}} B_{j+1,d-1}\right) - \lim_{x \to x^-}\left(\frac{t_{j+d+1}-x}{t_{j+d+1}-t_{j+1}} B_{j+1,d-1}\right)$$

$$= \frac{x-t_j}{t_{j+d}-t_j} J_x(B_{j,d-1}) + \frac{t_{j+d+1}-x}{t_{j+d+1}-t_{j+1}} J_x(B_{j+1,d-1})$$

$$J_x(B_{j,0}) = \begin{cases} 1 & x = t_j \\ -1 & x = t_{j+1} \\ 0 & \text{else} \end{cases}$$



furthermore,

$$\boxed{J_x(D^r B_{j,d}) = d\left(\frac{J_x(D^{r-1}B_{j,d-1})}{t_{j+d} - t_j} - \frac{J_x(D^{r-1}B_{j+1,d-1})}{t_{j+1+d} - t_{j+1}}\right)}$$

note that $\frac{0}{0} = 0$, $r \geq 1$ by differentiating the first derivative of $B_{j,d}$ recursive relation

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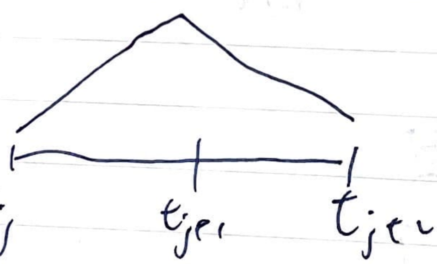
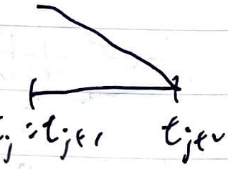
Lemma : if no knots of $t_j, \cdots, t_{j+d+1}$ occurs more than d times, B-spline $B_{j,d}$ is continuous.

proof : if $d=1$, no t_k occurs more than 1 time,

in other words,



note that



$J_{t_j}(B_{j,1}) = 1$

$\therefore B_{j,1}$ is continuous.

assume $d-1$ is true,

if no knots occurs more than $d-1$ times, so $B_{j,d-1}$ and $B_{j+1,d-1}$ are continuous, thus $B_{j,d}$ is also continuous.

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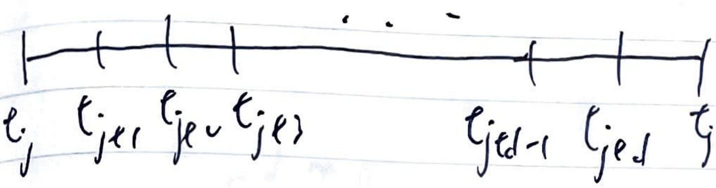
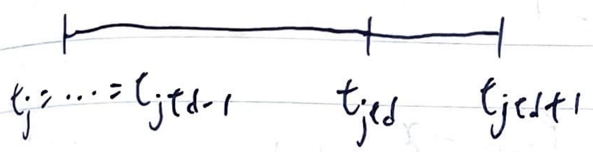
the only remaining case is ~~all~~ knots occur d times,



if $x = t_j$ (d times)

$$J_x(B_{j,d-1}) = 1$$



$$J_x(B_{j+1,d-1}) = 0 \quad \because \; d-1 \text{ times}$$

$$J_x(B_{j,d-1}) = 1 \quad \because \quad \underset{=t_j}{J_x}(B_{j,d-1}) = \frac{t_{j+d} - t_{j+1}}{t_{j+d} - t_{j+1}} J_{x=t_{j+1}}(B_{j+1,d-2})$$

$$= \frac{t_{j+d} - t_{j+2}}{t_{j+d} - t_{j+2}} J_{x=t_{j+2}}(B_{j+2,d-3})$$

$$\vdots$$

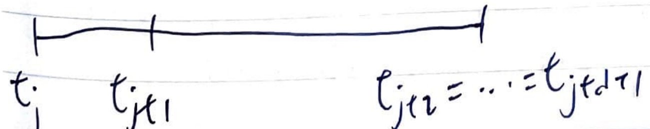
$$= \frac{t_{j+d} - t_{j+d-1}}{t_{j+d} - t_{j+d-1}} J_{x=t_{j+d-1}}(B_{j+d-1,0})$$

$$= 1$$

$$\therefore \; J_x(B_{j,d}) = \frac{x - t_j}{t_{j+d} - t_j} + 1 \cdot 0 = 0$$

Similarly, if $x = t_{j+d+1}$,

$$J_x(B_{j,d-1}) = 0$$



$$J_x(B_{j+1,d-1}) = \frac{t_{j+d} - t_j}{t_{j+d} - t_j} J_{x=t_{j+d}}(B_{j+1,d-2})$$

$$= J_{x=t_{j+2}}(B_{j+1,0}) = -1$$

$$\therefore J_x(B_{j,d}) = 1 \cdot 0 + \frac{t_{j+d+1} - x}{t_{j+d+1} - t_{j+1}} \cdot (-1) = 0$$

for $t_j < x < t_{j+d+1}$,

t_j —— $t_{j+1} = \cdots = t_{j+d}$ —— t_{j+d+1}

$J_x(B_{j,d-1}) = -1$, $J_x(B_{j+1,d-1}) = 1$

$$\therefore J_x(B_{j,d}) = \frac{x - t_j}{t_{j+d} - t_j}(-1) + \frac{t_{j+d+1} - x}{t_{j+d+1} - t_{j+1}} \cdot (1)$$

$$= \frac{-(t_{j+1} - t_j)(t_{j+d+1} - t_{j+1}) + (t_{j+d+1} - t_{j+1})(t_{j+1} - t_j)}{(t_{j+d} - t_j)(t_{j+d+1} - t_{j+1})}$$

$$= 0$$

//

Theorem : suppose $t_j, \ldots, t_{j+d+1}$, and define $B_{j,d}$. Suppose there are m occurence in knots, if $1 \le m \le d+1$, then $D^r B_{j,d}$ is continuous at x for $r = 0, \ldots, d-m$

proof: $r=0$, it is true by the previous lemma.

by induction, $r = d-m-1 \Rightarrow r = d-m$

$\therefore D^{d-m-1} B_{j,d-1}$ is continuous

$D^{d-m-1} B_{j+1,d-1}$ is continuous

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$$J_x(D^{d-m}B_{j,d}) = d\left(\frac{J_x(D^{d-m-1}B_{j,d-1})}{t_{j+d}-t_j} - \frac{J_x(D^{d-m-1}B_{j+1,d-1})}{t_{j+d+1}-t_{j+1}}\right)$$

$$= 0$$

$\therefore$ $D^{d-m}B_{j,d}$ is continuous.

for instance, $J_x(DB_{j,d}) = d\left(\frac{J_x(B_{j,d-1})}{t_{j+d}-t_j} - \frac{J_x(B_{j+1,d-1})}{t_{j+d+1}-t_{j+1}}\right)$

if $\exists$ $m = d-1$, $J_x(DB_{j,d}) = 0$,

$DB_{j,d}$ is continuous.

so when $d=1$, $DB_{j,1}$ is discontinuous, ($r=0$ only for continuous spline)
($m=1$ or $m=2$)

when $d=2$, $m=1$, $DB_{j,2}$ is continuous $\because$ $B_{j,1}$ is continuous.

$D^2B_{j,2}$ is discontinuous. $\because$ $DB_{j,1}$ is discontinuous.

$m=2$, $DB_{j,2}$ is discontinuous $\because$ $B_{j,1}$ is discontinuous.

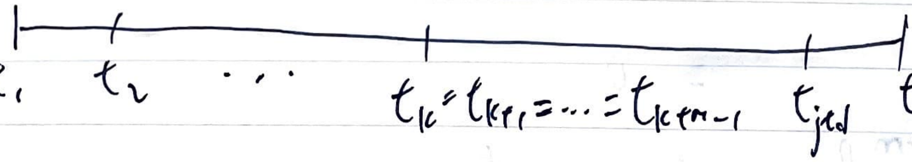
m at most $d+1$, $t_j=\dots=t_{j+d}$ — t_{j+d+1} or t_j — $t_{j+1}=\dots=t_{j+d+1}$

$d=1$	$d=2$
$r=0$	$r=0,1$

$\dots$ $r=0, \dots, d-m$

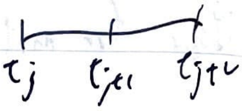
what if $D^r B_{j,d}$ when $r > d-m$, or $r = d-m+1$

($m = d+2$, all collapse to one point)



let's see $d=1$

$$J_x(DB_{j,1}) = \frac{J_x(B_{j,0})}{t_{j+1}-t_j} - \frac{J_x(B_{j+1,0})}{t_{j+2}-t_{j+1}}$$



$x = t_j$, $J_x(DB_{j,1}) = \frac{1}{t_{j+1}-t_j}$

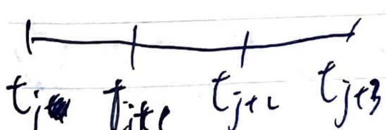
$x = t_{j+1}$, $J_x(DB_{j,1}) = \frac{-1}{t_{j+1}-t_j} - \frac{1}{t_{j+2}-t_{j+1}} = \frac{-t_{j+2}+t_j}{(t_{j+1}-t_j)(t_{j+2}-t_{j+1})}$

$$= \frac{t_{j+2}-t_j}{(t_j-t_{j+1})(t_{j+2}-t_{j+1})}$$

$x = t_{j+2}$, $J_x(DB_{j,1}) = \frac{1}{t_{j+2}-t_{j+1}}$

$d=2$,

$m=1$, $J_x(D^2B_{j,2}) = 2\left(\frac{J_x(DB_{j,1})}{t_{j+2}-t_j} - \frac{J_x(DB_{j+1,1})}{t_{j+3}-t_{j+1}}\right)$



$x = t_j$, $J_x(D^2B_{j,2}) = \frac{2}{\Delta_{10}\Delta_{20}}$ $\quad \Delta_{10} = t_{j+1}-t_j$

$\Delta_{kl} = t_{j+k}-t_{j+l}$

$x = t_{j+1}$, $J_x(D^2B_{j,2}) = \left(\frac{\Delta_{20}}{\Delta_{01}\Delta_{20}\Delta_{21}} - \frac{1}{\Delta_{21}\Delta_{31}}\right)2!$

$$= 2!\frac{\Delta_{20}\Delta_{31} - \Delta_{01}\Delta_{20}}{\Delta_{01}\Delta_{20}\Delta_{21}\Delta_{31}} = \frac{2!\Delta_{30}}{\Delta_{01}\Delta_{21}\Delta_{31}}$$

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$$x = t_{j+2},\quad J_x(D^2 B_{j,2}) = \left(\frac{1}{\Delta_{20}}\frac{1}{\Delta_{21}} - \frac{\Delta_{31}}{\Delta_{12}\Delta_{32}\Delta_{31}}\right)2!$$

$$= 2!\,\frac{\Delta_{32} - \Delta_{02}}{\Delta_{02}\Delta_{12}\Delta_{32}} = \frac{2!\,\Delta_{30}}{\Delta_{02}\Delta_{12}\Delta_{32}}$$

$$x = t_{j+3},\quad J_x(D^2 B_{j,2}) = -\frac{2!}{\Delta_{31}\Delta_{32}} = -\frac{2!}{\Delta_{13}\Delta_{23}}$$

$$m = 2,\quad J_x(D B_{j,2}) = 2\left(\frac{J_x(B_{j,1})}{t_{j+2} - t_j} - \frac{J_x(B_{j+1,1})}{t_{j+3} - t_{j+1}}\right)$$

t_j　　$t_{j+1} = t_{j+2}$　　t_{j+3}

$$= \frac{-2}{t_{j+2} - t_j} - \frac{2}{t_{j+3} - t_{j+1}}$$

$$= 2!\,\frac{t_{j+1} + t_j - t_{j+2} - t_{j+3}}{(t_{j+2} - t_j)(t_{j+3} - t_{j+1})}$$

$$= \frac{2!\,\Delta_{30}}{\Delta_{01}\Delta_{31}}$$

t_j　　$t_{j+1} = t_{j+2} = t_{j+3}$

$J_x(B_{j,2}) = -1$

$t_j = t_{j+1}$　　t_{j+2}　　t_{j+3}

$$J_x(B_{j,2}) = \frac{2}{\Delta_{20}}$$

t_j　　t_{j+1}　　$t_{j+2} = t_{j+3}$

$$J_x(B_{j,2}) = \frac{-1}{\Delta_{31}} = \frac{1}{\Delta_{13}}$$

$d = 3,$

$m = 1$

$$J_x(D^3 B_{j,3}) = 3\left(\frac{J_x(D^2 B_{j,2})}{t_{j+3} - t_j} - \frac{J_x(D^2 B_{j+1,2})}{t_{j+4} - t_{j+1}}\right)$$

t_j　t_{j+1}　t_{j+2}　t_{j+3}　t_{j+4}

$$x = t_j,\quad J_x(D^3 B_{j,3}) = 3!\,\frac{1}{\Delta_{30}}\,\frac{1}{\Delta_{10}\Delta_{20}} = 3!\,\frac{1}{\Delta_{10}\Delta_{20}\Delta_{30}}$$

$$x = t_{j+1},\quad J_x(D^3 B_{j,3}) = \left(\frac{1}{\Delta_{30}}\,\frac{\Delta_{30}}{\Delta_{01}\Delta_{21}\Delta_{31}} - \frac{1}{\Delta_{41}}\,\frac{1}{\Delta_{21}\Delta_{31}}\right)3!$$

$$= 3!\,\frac{\Delta_{41} - \Delta_{01}}{\Delta_{01}\Delta_{21}\Delta_{31}\Delta_{41}} = \frac{3!\,\Delta_{40}}{\Delta_{01}\Delta_{21}\Delta_{31}\Delta_{41}}$$

$x=t_{j+2}$, $J_x(D^3B_{j,3}) = 3!\left(\frac{\Delta_{30}}{\Delta_{30}\Delta_{02}\Delta_{12}\Delta_{32}} - \frac{\Delta_{41}}{\Delta_{41}\Delta_{12}\Delta_{32}\Delta_{42}}\right)$

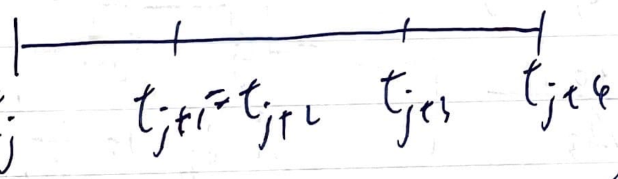
$= 3!\,\frac{\Delta_{42}-\Delta_{02}}{\Delta_{02}\Delta_{12}\Delta_{32}\Delta_{42}} = \frac{3!\,\Delta_{40}}{\Delta_{02}\Delta_{12}\Delta_{32}\Delta_{42}}$

$x=t_{j+3}$, $J_x(D^3B_{j,3}) = 3!\left(\frac{-1}{\Delta_{30}\Delta_{03}\Delta_{23}} - \frac{1}{\Delta_{13}\Delta_{23}\Delta_{43}}\right)$

$= 3!\,\frac{\Delta_{43}-\Delta_{03}}{\Delta_{03}\Delta_{13}\Delta_{23}\Delta_{43}} = 3!\,\frac{\Delta_{40}}{\Delta_{03}\Delta_{13}\Delta_{23}\Delta_{43}}$

$x=t_{j+4}$, $J_x(D^3B_{j,3}) = 3!\,\frac{1}{\Delta_{41}}\,\frac{-1}{\Delta_{24}\Delta_{34}} = \frac{3!}{\Delta_{14}\Delta_{24}\Delta_{34}}$

$m=2$



$x=t_{j+1}$, $J_x(D^2B_{j,3}) = \cancel{3!\left(\frac{1}{\Delta_{01}\Delta_{31}}\right)}$

$3!\left(\frac{1}{\Delta_{01}\Delta_{31}} - \frac{1}{\Delta_{31}\Delta_{41}}\right) = 3!\,\frac{\Delta_{40}}{\Delta_{01}\Delta_{31}\Delta_{41}}$

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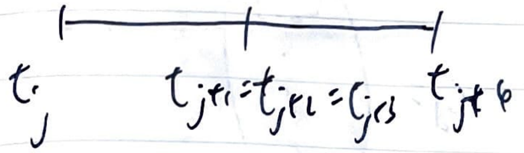
$m = 3$,



$x = t_{j+1}$,

$$J_x(DB_{j,3}) = 3\left(\frac{J_x(B_{j,2})}{t_{j+3}-t_j} - \frac{J_x(B_{j+1,2})}{t_{j+4}-t_{j+1}}\right)$$

$$= 3\left(\frac{-1}{\Delta_{10}} - \frac{1}{\Delta_{41}}\right)$$

$$= 3\,\frac{\Delta_{40}}{\Delta_{01}\,\Delta_{41}}$$

in general, as long as multiplicity is ~~m~~ in between,

$$\boxed{J_x(D^{d-m+1}B_{j,d}) = \frac{d!}{(m-1)!}(t_{j+d+1}-t_j)\prod_{\substack{k=j \\ k\neq \text{multiplicity}}}^{j+d+1}\frac{1}{t_k-x}}$$

if m multiplicity in the middle of knots,

$$J_x(D^{d-m+1}B_{j,d}) = \frac{d}{(m-1)!}\cdot(d-1)!\left(\prod_{\substack{k=j \\ k\neq \text{mult.}}}^{j+d}\frac{1}{t_k-x} - \prod_{\substack{k=j+1 \\ k\neq \text{mult.}}}^{j+d+1}\frac{1}{t_k-x}\right)$$

simple proof

$$= \frac{d!}{(m-1)!}\left(\frac{1}{t_j-x} - \frac{1}{t_{j+d+1}-x}\right)\prod_{\substack{k=j+1 \\ k\neq \text{mult.}}}^{j+d}\frac{1}{t_k-x}$$

$$= \frac{d!}{(m-1)!}(t_{j+d+1}-t_j)\prod_{\substack{k=j \\ k\neq \text{mult.}}}^{j+d+1}\frac{1}{t_k-x}$$

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Evaluation of derivative:

$$B_d(x) = (B_{\mu-d,d} \cdots B_{\mu,d})\, C_d, \quad C_d = \begin{pmatrix} C_{\mu-d} \\ \vdots \\ C_\mu \end{pmatrix}$$

$$D^r f(x) = \frac{d!}{(d-r)!} R_1(x) \cdots R_{d-r}(x)\, DR_{d-r+1} \cdots DR_d \cdot C_d$$

compute from right to left,

$$C_d^{(0)} = C_d, \qquad C_{d-1}^{(1)} = DR_d\, C_d^{(0)}$$

$$C_{d-2}^{(2)} = DR_{d-1}\, C_{d-1}^{(1)} \qquad \therefore C_{k-1}^{(d-k+1)} = DR_k\, C_k^{(d-k)}$$

$$\vdots$$

$$C_{d-r}^{(r)} = DR_{d-r+1}\, C_{d-r+1}^{(r-1)}$$

Ex.,

$$C_{d-1}^{(1)} = \begin{pmatrix} \dfrac{-C_{\mu-d}}{t_{\mu+1} - t_{\mu+1-d}} + \dfrac{C_{\mu-d+1}}{t_{\mu+1} - t_{\mu+1-d}} \\ \vdots \\ \dfrac{-C_k}{t_{d+k+1} - t_{k+1}} + \dfrac{C_{k+1}}{t_{d+k+1} - t_{k+1}} \\ \vdots \\ \dfrac{-C_{\mu-1}}{t_{\mu+d} - t_\mu} + \dfrac{C_\mu}{t_{\mu+d} - t_\mu} \end{pmatrix}$$

$$C_{d-r-1}^{(r)} = R\, C_{d-r}^{(r)}$$

$$C_{d-r-2}^{(r)} = R_{d-r-1}\, C_{d-r-1}^{(r)} \qquad \therefore C_{k-1}^{(r)} = R_k\, C_k^{(r)}$$

$$\vdots$$

$$C_0^{(r)} = R_1\, C_1^{(r)}$$

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compute from left to right,

$$B_0 = 1$$

$$B_1 = B_0 R_1$$

$$\vdots \qquad \therefore B_K = B_{K-1} R_K$$

$$B_{d-r} = B_{d-r-1} R_{d-r}$$

$$D B_{d-r+1} = B_{d-r} D R_{d-r+1}$$

$$D^2 B_{d-r+2} = D B_{d-r+1} D R_{d-r+2} \qquad \therefore D^{k-d+r} B_K = D B_{K-1} D R_K$$

$$\vdots$$

$$D^r B_d = D^{r-1} B_{d-1} D R_d$$

$$= \left(D^r B_{\mu-d,d} \quad \cdots \quad D^r B_{\mu,d} \right)$$

$$D^{k-d+r} B_K = \left(D^{k-d+r-1} B_{\mu-k+1,d} \quad \cdots \quad D^{k-d+r-1} B_{\mu,d} \right) \begin{pmatrix} \frac{-1}{t_{\mu+1} - t_{\mu+1-k}} & \frac{1}{t_{\mu+1} - t_{\mu+1-k}} & & \\ & \ddots & \ddots & \\ & & \frac{1}{t_{\mu+k} - t_{\mu}} & \frac{1}{t_{\mu+k} - t_{\mu}} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{D^{k-d+r-1} B_{\mu-k+1}}{t_{\mu+1} - t_{\mu+1-k}} \\ \vdots \\ \frac{D^{k-d+r-1} B_{l}}{t_{l-k} - t_{l-2k}} - \frac{D^{k-d+r-1} B_{l+1}}{t_{l+k+1} - t_{l-2k+1}} \\ \vdots \end{pmatrix}^T$$

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Knot insertion

if $\tau \subseteq t$, $S_{d,\tau} \subseteq S_{d,t}$

$$f = \sum_{j=1}^{n} c_j B_{j,d,\tau} = \sum_{j=1}^{m} b_j B_{j,d,t}$$

$$B_{j,d,\tau} = \sum_{i=1}^{m} \alpha_{j,d}(i) B_{i,d,t}$$

$$\therefore B_\tau^T = B_t^T A, \quad A = \begin{pmatrix} \alpha_{1,d}(1) & & \alpha_{n,d}(1) \\ \vdots & & \vdots \\ \alpha_{1,d}(i) & \cdots & \alpha_{n,d}(i) \\ \vdots & & \vdots \\ \alpha_{1,d}(m) & & \alpha_{n,d}(m) \end{pmatrix}$$

$A \in M_{mn}$

$$f = \sum_{j=1}^{n} c_j B_{j,d,\tau}$$

$$\cancel{= \sum_{j=1}^{n} c_j \alpha_{j,d}(i) B_{i,d,t}}$$

$$= \sum_{j=1}^{n} c_j \sum_{i=1}^{m} \alpha_{j,d}(i) B_{i,d,t}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} c_j \alpha_{j,d}(i) B_{i,d,t}$$

$$= \sum_{j=1}^{m} \sum_{i=1}^{n} c_i \alpha_{i,d}(j) B_{j,d,t}$$

$$= \sum_{j=1}^{m} \left(\sum_{i=1}^{n} c_i \alpha_{i,d}(j) \right) B_{j,d,t}$$

$$\therefore b_j = \sum_{i=1}^{n} c_i \alpha_{i,d}(j).$$

$$b = Ac$$

or $f = B_t^T b = B_\tau^T c = B_t^T A c$

$$\therefore b = Ac$$

A is knot insertion matrix from τ to t,

$$\alpha_{j,d}(i) = \alpha_{j,d,\tau,t}(i)$$

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Example :

$\tau = (0, 1, 2) = (\tau_i)_{i=1}^{3}$

$t = (0, \frac{1}{2}, 1, \frac{3}{2}, 2) = (t_i)_{i=1}^{5}$

$B_{i,d,\tau} \Leftarrow B_{j,d,t}$

d=0

$S_{d,\tau} = \text{span}\{B_{1,0,\tau}, B_{2,0,\tau}\}$ $S_{d,t} = \text{span}\{B_{1,0,t}, B_{2,0,t}, B_{3,0,t}, B_{4,0,t}\}$

$\therefore B_{1,0,\tau} = B_{1,0,t} + B_{2,0,t}$

$B_{2,0,\tau} = B_{3,0,t} + B_{4,0,t}$

d=1

$B_1(0,1,2) = B_{1,1,\tau}$

$= \frac{x-0}{1-0} B(0,1) + \frac{2-x}{2-1} B(1,2)$

$= x B(0,1) + (2-x) B(1,2)$

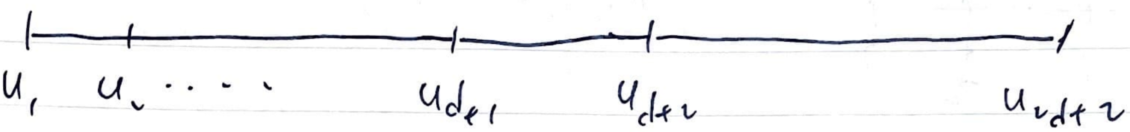
$B_1(0, \frac{1}{2}, 1) = \frac{x}{1/2} B(0, \frac{1}{2}) + \frac{(1-x)}{1/2} B(\frac{1}{2}, 1)$

$B(\frac{1}{2}, 1, \frac{3}{2}) = \frac{x-\frac{1}{2}}{\frac{1}{2}} B(\frac{1}{2}, 1) + \frac{\frac{3}{2}-x}{1/2} B(1, \frac{3}{2})$

$B(1, \frac{3}{2}, 2) = \frac{x-1}{1/2} B(1, \frac{3}{2}) + \frac{2-x}{\frac{1}{2}} B(\frac{3}{2}, 2)$

$\frac{1}{2} B(0, \frac{1}{2}, 1) + B(\frac{1}{2}, 1, \frac{3}{2}) + \frac{1}{2} B(1, \frac{3}{2}, 2) = x B(0, \frac{1}{2}) + x B(\frac{1}{2}, 1)$
$+ (2-x) B(1, \frac{3}{2}) + (2-x) B(\frac{3}{2}, 2)$

Let's rewrite, $B_{d,v}^T = B_{d,u}^T M_{u,v}^d$ for conversion of knots $(u_i)_{i=1}^{2d+2}$ to $(v_i)_{i=1}^{2d+2}$, $u_{d+1} < u_{d+2}$, $v_{d+1} < v_{d+2}$, $M_{u,v}^d \in M_{d+1,d+1}$



recall that $B_{d,u}^T P_{d,u} = (y-x)^d$

$$B_{d,u}^T(x) = \left(B_{1,d,u}(x)\ \ B_{2,d,u}(x)\ \cdots\ B_{d+1,d,u}(x)\right)$$

$$P_{d,u}(y) = \left(P_{1,d,u}(y)\ \ P_{2,d,u}(y)\ \cdots\ P_{d+1,d,u}(y)\right)^T$$

$$P_{i,d,u}(y) = (y-u_{i+1})(y-u_{i+2})\cdots(y-u_{i+d})$$

~~note~~ that recall also $P_{d-1,u}(y)(y-a) = R_d^{d+1}(a)\, P_{d,u}(y)$ $\quad R_{d,u}^{d+1}(a) \in M_{d,d+1}$, $R_{1,u}^{d+1}(a) \in M_{1,2}$

$$\therefore (y-v_{i+1})(y-v_{i+2})\cdots(y-v_{i+d})$$

$$= R_{1,u}^{d+1}(v_{i+1})\, R_{2,u}^{d+1}(v_{i+2}) \cdots R_{d,u}^{d+1}(v_{i+d})\, P_{d,u}(y)$$

~~ ~~ note that this is equal to

$$(y-v_{i+1})(y-v_{i+2})\cdots(y-v_{i+d}) = P_{i,d,v}(y)$$

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$$\therefore P_{i,d,v}(y) = R^{d+1}_{1,u}(v_{i+1}) \cdots R^{d+1}_{d,u}(v_{i+d}) P_{d,u}(y)$$

$$P_{d,v}(y) = \mathbb{R}^{d+1}_{d,u}(v) P_{d,u}(y)$$

where
$$\mathbb{R}_u(v) = \begin{pmatrix} R^{d+1}_{1,u}(v_2) & R^{d+1}_{2,u}(v_3) & & R^{d+1}_{d,u}(v_{d+1}) \\ R^{d+1}_{1,u}(v_3) & R^{d+1}_{2,u}(v_4) & \cdots & R^{d+1}_{d,u}(v_{d+2}) \\ \vdots & & & \vdots \\ R^{d+1}_{1}(v_{d+2}) & R^{d+1}_{2,u}(v_{d+3}) & & R^{d+1}_{d,u}(v_{2d+1}) \end{pmatrix}$$

$$\therefore P_{1,d,v}(y) = (y-v_2)\cdots(y-v_{d+1}) = R^{d+1}_{1,u}(v_2)\cdots R^{d+1}_{d,u}(v_{d+1}) P_{d,u}(y)$$

$$P_{2,d,v}(y) = (y-v_3)\cdots(y-v_{d+2}) = R^{d+1}_{1,u}(v_3)\cdots R^{d+1}_{d,u}(v_{d+2}) P_{d,u}(y)$$

$$\vdots$$

$$P_{d+1,d,v}(y) = (y-v_{d+2})\cdots(y-v_{2d+1}) = R^{d+1}_{1,u}(v_{d+2})\cdots R^{d+1}_{d,u}(v_{2d+1}) P_{d,u}(y)$$

note that $R^{d+1}_{1,u}(v_{i+1}) \cdots R^{d+1}_{d,u}(v_{i+d})$ is a $1 \times d+1$ matrix.

~~note that~~ $\cancel{P_{d,v}^T B_{d,v}^T = P_{d,v}^T B_{d,u}^T M^d_{u,v}}$

$\cancel{(y-x)^d =}$

hence,
$$B^T_{d,v} P_{d,v} = B^T_{d,u} M^d_{u,v} P_{d,v}$$

$$(y-x)^d = B^T_{d,u} M^d_{u,v} P_{d,v}$$

$$= B^T_{d,u} M^d_{u,v} \mathbb{R}^{d+1}_{d,u} P_{d,u}$$

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$$(y-x)^d B^{-T}_{d,u} P^{-1}_{d,u} = M^{d}_{u,v} \mathbb{R}^{d+1}_{d,u}$$

$$\therefore M^{d}_{u,v} \mathbb{R}^{d+1}_{d,u} = (y-x)^d \left(P_{d,u} B^{T}_{d,u} \right)^{-1}$$

$$P_{d,v}(y) = \mathbb{R}^{d+1}_{d,u}(v) P_{d,u}(y)$$

$$\because P_{d,u}(y) = (y-x)^d B^{-T}_{d,u}(x), \quad P_{d,v}(y) = (y-x)^d B^{-T}_{d,v}(x)$$

$$\therefore B^{-T}_{d,v}(x) = \mathbb{R}^{d+1}_{d,u}(v) B^{-T}_{d,u}(x)$$

$$B^{T}_{d,u}(x) = B^{T}_{d,v}(x) \mathbb{R}^{d+1}_{d,u}(v)$$

$$\therefore \boxed{M^{d}_{v,u} = \mathbb{R}^{d+1}_{d,u}(v)}$$

NOTE THAT u,v are reversed!! in LHS and RHS

~~Example:~~

let us consider t be a refined knots of τ,

$$(t_i)_{i=1}^{m+d+1} \qquad (\tau_i)_{i=1}^{n+d+1}$$

$$B^{T}_{\tau} = B^{T}_{t} A$$

~~(note that it is reverse from the previous definition)~~.

$$B_{j,d,\tau} = \sum_{i=1}^{m} \alpha_{j,d}(i) B_{i,d,t}$$

the reason that we use $t \to \tau$ because in the interval below?

(cancelled!)

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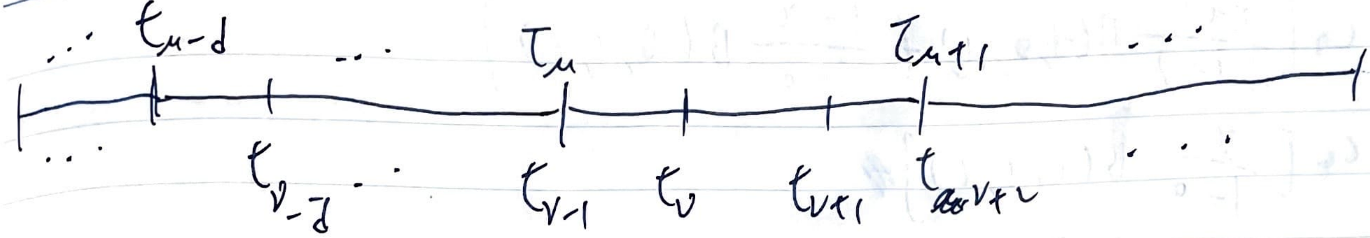

$$[t_{\nu-d}, t_{\nu+1}] \subseteq [\tau_{\mu-d}, \tau_{\mu+1}]$$

τ region always include t region, when we consider $[t_\nu, t_{\nu+1})$

$$A = \begin{pmatrix} \alpha_{1,d}(1) & & \alpha_{n,d}(1) \\ \vdots & \cdots & \vdots \\ \alpha_{1,d}(m) & & \alpha_{n,d}(m) \end{pmatrix}$$

$$b = Ac, \quad f = \sum_j c_j B_{j,d,\tau} = \sum_j b_j B_{j,d,t}$$

Examples:

$$\tau = (-1, -1, -1, 0, 1, 1, 1)$$

$$t = (-1, -1, -1, -\tfrac{1}{2}, 0, \tfrac{1}{2}, 1, 1, 1)$$

$$f = \sum_{i=1}^{4} c_i B_{i,2,\tau}$$

$$= c_1 B(-1,-1,-1,0) + c_2 B(-1,-1,0,1) + c_3 B(-1,0,1,1)$$

$$+ c_4 B(0,1,1,1)$$

$$= c_1 \frac{0-x}{0-(-1)} B(-1,-1,0) + c_2 \left[\frac{x+1}{0-(-1)} B(-1,-1,0) + \frac{1-x}{1-(-1)} B(-1,0,1) \right]$$

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$$+ c_3\left[\frac{x+1}{1-(-1)} B(-1,0,1) + \frac{1-x}{1-0} B(0,1,1)\right]$$

$$+ c_4\left[\frac{x}{1-0} B(0,1,1)\right]$$

$$= c_1 \cdot (-x) \cdot \frac{0-x}{0-(-1)} B(-1,0) + c_2\left[(x+1) \cdot \frac{0-x}{0-(-1)} B(-1,0)\right.$$

$$\left.+ \frac{1-x}{2}\left[(x+1)B(-1,0) + (1-x)B(0,1)\right]\right] + c_3\left[\frac{x+1}{2}\left[(x+1)B(-1,0)\right.\right.$$

$$\left.\left.+ (1-x)B(0,1)\right] + (1-x)\left[x \cdot B(0,1)\right]\right] + c_4 \cdot x \cdot x B(0,1)$$

$$= \left(c_1 x^2 - c_2 x(x+1) + c_2 \frac{1-x^2}{2} + c_3 \frac{(x+1)^2}{2}\right) B(-1,0)$$

$$+ \left(c_2 \frac{(1-x)^2}{2} + c_3 \frac{1-x^2}{2} + c_3 (1-x)x + c_4 x^2\right) B(0,1)$$

$$= \left[x^2 c_1 + \frac{(1-3x)(1+x)}{2} c_2 + \frac{(1+x)^2}{2} c_3\right] B(-1,0)$$

$$+ \left[\frac{(1-x)^2}{2} c_2 + \frac{(1+3x)(1-x)}{2} c_3 + x^2 c_4\right] B(0,1)$$

$$f = \sum_{i=1}^{6} b_i B_{i,2,t}$$

$$= b_1 B(-1,-1,-1,-\tfrac{1}{2}) + b_2 B(-1,-1,-\tfrac{1}{2},0) + b_3 B(-1,-\tfrac{1}{2},0,\tfrac{1}{2}) + b_4 B(-\tfrac{1}{2},0,\tfrac{1}{2},1)$$

$$+ b_5 B(0,\tfrac{1}{2},1,1) + b_6 B(\tfrac{1}{2},1,1,1)$$

$$= -b_1(1+2x) B(-1,-1,-\tfrac{1}{2}) + b_2\left[2(1+x) B(-1,-1,-\tfrac{1}{2}) + (-x) B(-1,-\tfrac{1}{2},0)\right]$$

$$+ b_3\left[(1+x) B(-1,-\tfrac{1}{2},0) + (\tfrac{1}{2}-x) B(-\tfrac{1}{2},0,\tfrac{1}{2})\right] + b_4\left[(x+\tfrac{1}{2}) B(-\tfrac{1}{2},0,\tfrac{1}{2})\right.$$

$$\left.+ (1-x) B(0,\tfrac{1}{2},1)\right] + b_5\left[x B(0,\tfrac{1}{2},1) + 2(1-x) B(\tfrac{1}{2},1,1)\right]$$

$$+ b_6\left[(2x-1) B(\tfrac{1}{2},1,1)\right]$$

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$$= b_1 (1+2x)^2 B(-1,-\tfrac{1}{2}) + b_2 \Big[-2(1+x)(1+2x) B(-1,-\tfrac{1}{2}) + 2(-x)(1+x) B(-1,-\tfrac{1}{2})$$

$$+ 2x^2 B(-\tfrac{1}{2},0) \Big] + b_3 \Big[2(1+x)^2 B(-1,-\tfrac{1}{2}) - 2x(1+x) B(-\tfrac{1}{2},0)$$

$$+ 2(\tfrac{1}{4}-x^2) B(-\tfrac{1}{2},0) + 2(\tfrac{1}{2}-x)^2 B(0,\tfrac{1}{2}) \Big] + b_4 \Big[2(x+\tfrac{1}{2})^2 B(-\tfrac{1}{2},0)$$

$$+ 2(\tfrac{1}{4}-x^2) B(0,\tfrac{1}{2}) + 2x(1-x) B(0,\tfrac{1}{2}) + 2(1-x)^2 B(\tfrac{1}{2},1) \Big] + b_5 \Big[$$

$$2x^2 B(0,\tfrac{1}{2}) + 2x(1-x) B(\tfrac{1}{2},1) + 4(1-x)(x-\tfrac{1}{2}) B(\tfrac{1}{2},1) \Big]$$

$$+ b_6 (2x-1)^2 B(\tfrac{1}{2},1)$$

$$= \Big[(1+2x)^2 b_1 - 2(1+x)(1+2x) b_2 + 2(-x)(1+x) b_2 + 2(1+x)^2 b_3 \Big] B(-1,-\tfrac{1}{2})$$

$$+ \Big[2x^2 b_2 - 2x(1+x) b_3 + 2(\tfrac{1}{4}-x^2) b_3 + 2(x+\tfrac{1}{2})^2 b_4 \Big] B(-\tfrac{1}{2},0)$$

$$+ \Big[2(\tfrac{1}{2}-x)^2 b_3 + 2(\tfrac{1}{4}-x^2) b_4 + 2x(1-x) b_4 + 2x^2 b_5 \Big] B(0,\tfrac{1}{2})$$

$$+ \Big[2(1-x)^2 b_4 + 2x(1-x) b_5 + 4(1-x)(x-\tfrac{1}{2}) b_5 + (2x-1)^2 b_6 \Big] B(\tfrac{1}{2},1)$$

$$= \Big[(1+2x)^2 b_1 - 2(1+x)(1+3x) b_2 + 2(1+x)^2 b_3 \Big] B(-1,-\tfrac{1}{2})$$

$$+ \Big[2x^2 b_2 + \frac{1-4x-8x^2}{2} b_3 + 2(x+\tfrac{1}{2})^2 b_4 \Big] B(-\tfrac{1}{2},0)$$

$$+ \Big[2(\tfrac{1}{2}-x)^2 b_3 + \frac{1+4x-8x^2}{2} b_4 + 2x^2 b_5 \Big] B(0,\tfrac{1}{2})$$

$$+ \Big[2(1-x)^2 b_4 + 2(1-x)(3x-1) b_5 + (2x-1)^2 b_6 \Big] B(\tfrac{1}{2},1)$$

by comparison of coefficients

$B(-1,0)$: $x^2 c_1 + (\tfrac{1}{2} - x - \tfrac{3}{2}x^2) c_2 + (\tfrac{1}{2} + x + \tfrac{x^2}{2}) c_3$

$$= (c_1 - \tfrac{3}{2}c_2 + \tfrac{1}{2}c_3) x^2 + (c_3 - c_2) x + \frac{c_2 + c_3}{2}$$

$B(0,1))$: $(\tfrac{1}{2} - x + \tfrac{x^2}{2}) c_2 + (\tfrac{1}{2} + x - \tfrac{3x^2}{2}) c_3 + x^2 c_4$

$$= (\tfrac{1}{2}c_2 - \tfrac{3}{2}c_3 + c_4) x^2 + (c_3 - c_2) x + \frac{c_2 + c_3}{2}$$

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$$B(-1,-\tfrac{1}{2}) : (1+4x+4x^2)b_1 - 2(1+4x+3x^2)b_2 + 2(1+2x+x^2)b_3$$
$$= (4b_1 - 6b_2 + 2b_3)x^2 + (4b_1 - 8b_2 + 4b_3)x + b_1 - 2b_2 + 2b_3$$

$$B(-\tfrac{1}{2},0) : 2x^2 b_2 + \frac{1-4x-8x^2}{2} b_3 + 2(x^2+x+\tfrac{1}{4})b_4$$
$$= (2b_2 - 4b_3 + 2b_4)x^2 + (-2b_3 + 2b_4)x + \frac{b_3+b_4}{2}$$

$$B(0,\tfrac{1}{2}) : 2(\tfrac{1}{4} - x + x^2)b_3 + \frac{1+4x-8x^2}{2} b_4 + 2x^2 b_5$$
$$= (2b_3 - 4b_4 + 2b_5)x^2 + 2(b_4 - b_3)x + \frac{b_3+b_4}{2}$$

$$B(\tfrac{1}{2},1) : 2(1-2x+x^2)b_4 + 2(-1+4x-3x^2)b_5 + (4x^2-4x+1)b_6$$
$$= (2b_4 - 6b_5 + 4b_6)x^2 + 4(-b_4 + 2b_5 - b_6)x + 2b_4 - 2b_5 + b_6$$

$$B_t(-1,-\tfrac{1}{2}) \sim B_\tau(-1,0)$$

$$\begin{cases} C_1 - \frac{3}{2}C_2 + \frac{1}{2}C_3 = 2(2b_1 - 3b_2 + b_3) & ① \\ C_3 - C_2 = 4(b_1 - 2b_2 + b_3) & ② \\ \frac{C_2 + C_3}{2} = b_1 - 2b_2 + 2b_3 & ③ \end{cases}$$

$2① - ②$: $4b_1 - 4b_2 = 2C_1 - 2C_2$

$① - ③$: $3b_1 - 4b_2 = C_1 - 2C_2$, $\therefore b_1 = C_1$

$b_2 = \frac{1}{2}(C_1 + C_2)$

from ③, $2b_3 + C_1 - (C_1 + C_2) = \frac{C_2+C_3}{2}$, $\therefore b_3 = \frac{3C_2 + C_3}{4}$

$$B_t(\tfrac{1}{2},1) \sim B_\tau(0,1)$$

$$\begin{cases} \frac{1}{2}C_2 - \frac{3}{2}C_3 + C_4 = 2(b_4 - 3b_5 + 2b_6) & ① \\ C_3 - C_2 = 4(-b_4 + 2b_5 - b_6) & ② \\ \frac{C_2 + C_3}{2} = 2b_4 - 2b_5 + b_6 & ③ \end{cases}$$

B-spline

②+2①: $4b_6 - 4b_5 = c_4 - 2c_3$,
①−③: $3b_6 - 4b_5 = c_4 - 2c_3$, $\therefore$ $b_6 = c_4$, $b_5 = \frac{1}{2}(c_3 + c_4)$

from ③, $\frac{c_2 + c_3}{2} = 2b_4 - (c_3 + c_4) + c_4$, $\therefore$ $b_4 = \frac{c_2 + 3c_3}{4}$

$b = Ac$

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}$$

checking the answer,

$B_\tau(-\frac{1}{2}, 0) \sim B_t(-1, 0)$

$$\begin{cases} c_1 - \frac{3}{2}c_2 + \frac{1}{2}c_3 = 2(b_2 - 2b_3 + b_4) & ① \\ c_3 - c_2 = 2(b_4 - b_3) & ② \\ \frac{c_2 + c_3}{2} = \frac{b_3 + b_4}{2} & ③ \end{cases}$$

from ② and ③, $b_3 = \frac{3c_2 + c_3}{4}$, $b_4 = \frac{c_2 + 3c_3}{4}$,

$\therefore$ $2b_2 - (3c_2 + c_3) + \frac{1}{2}(c_2 + 3c_3) = c_1 - \frac{3}{2}c_2 + \frac{1}{2}c_3$

$b_2 = \frac{c_1 + c_2}{2}$

let's return to the general case

$B_t^T = B_\tau^T A$, $A = \mathbb{R}^{\cancel{\#\#}\, m'}_{d,\tau}(t)$